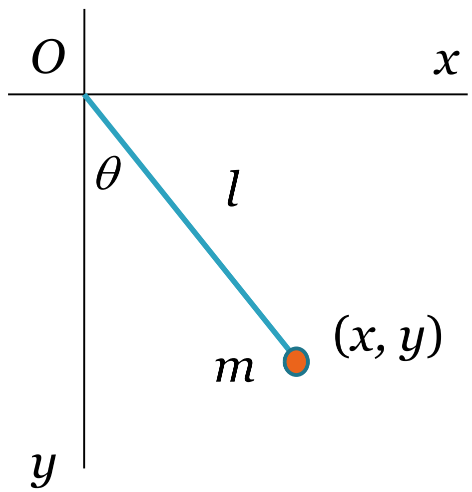
PHYS 705: Classical Mechanics

Simple Examples using
Lagrange Equations

Simple Pendulum



We can describe the pendulum by the Cartesian coordinates (x, y) as shown with the constraint:

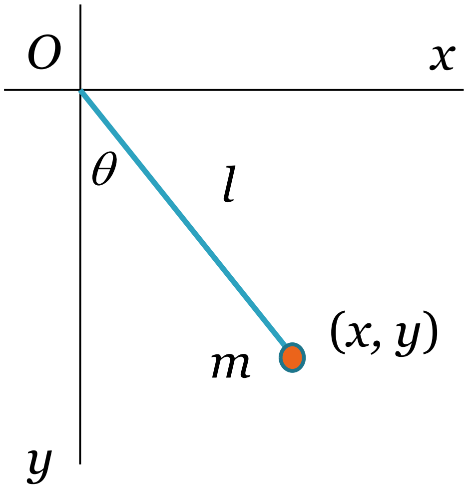
$$f(x, y) = x^2 + y^2 - l^2 = 0$$

➡ One can effectively describe this pendulum with a fixed length and one *independent generalized coordinate* θ.

The Lagrange's Equations of Motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad \text{with} \qquad L = T - U$$

Simple Pendulum



With one independent generalized coordinate θ, we can express (x, y) in term of the chosen generalized coordinate θ :

$$x = l\sin\theta \qquad \dot{x} = l\cos\theta\dot{\theta}$$

$$y = l\cos\theta \qquad \dot{y} = -l\sin\theta\dot{\theta}$$

(constraint is implicit here)

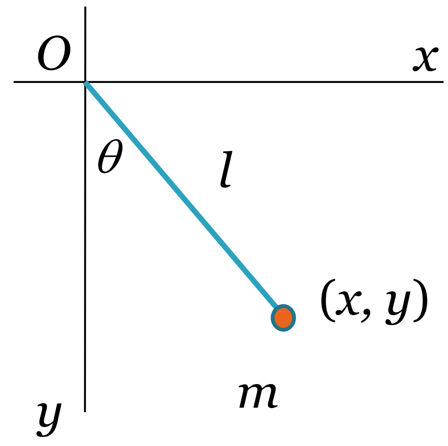
Simple Pendulum

Next, we form the Lagrangian, $L = T - U$:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$T = \frac{1}{2}m\left(l^2\cos^2\theta\dot{\theta}^2 + l^2\sin^2\theta\dot{\theta}^2\right)$$

$$= \frac{1}{2}ml^2\dot{\theta}^2 \qquad \text{also}\left(T = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(ml^2\right)\dot{\theta}^2\right)$$



Coord Transformation:

$$x = l\sin\theta \qquad \dot{x} = l\cos\theta\dot{\theta}$$

$$y = -l\cos\theta \qquad \dot{y} = l\sin\theta\dot{\theta}$$

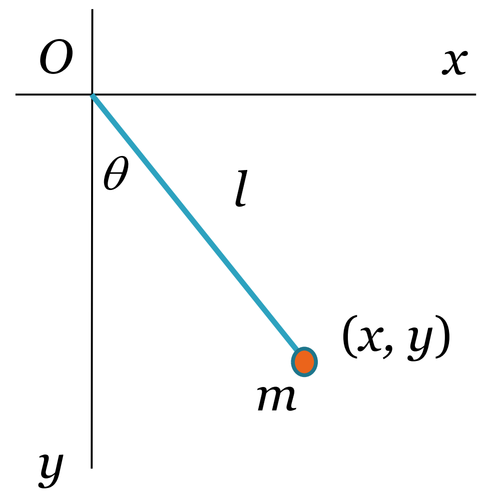
(constraint is implicit here)

Now, define U=0 at y=0. Then, $U = mgy = -mgl\cos\theta$

Finally, $L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

Simple Pendulum

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$



The EL equation is, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

So, we have,

$$\frac{d}{dt}\left(m l^2 \dot{\theta}\right) + mgl\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

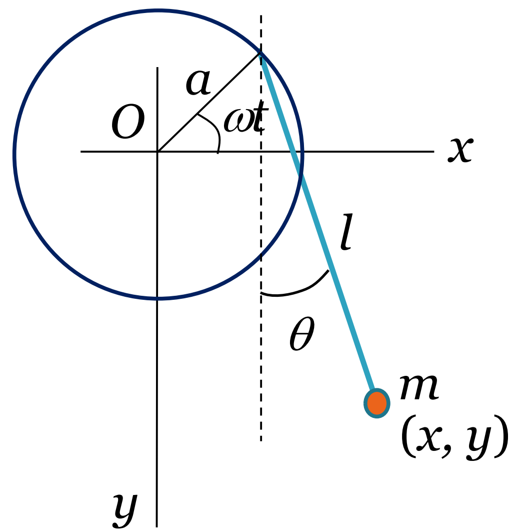
This is the familiar pendulum equation with SHO when θ is small:

$$\sin\theta \simeq \theta \quad \rightarrow \quad \ddot{\theta} + \frac{g}{l}\theta = 0$$

Pendulum with a Rotating Support

The support of the pendulum moves in a circle of radius a and constant angular velocity ω.

- With the given constraint, this problem will be very cumbersome if we try to solve for the EOM using the 2nd law directly.
- Here, we will use the Lagrangian Formalism,



Since ω is fixed, one can effectively describe this problem with just one generalized coordinate θ.

$$x = a\cos(\omega t) + l\sin\theta \qquad \dot{x} = -a\omega\sin(\omega t) + l\cos\theta\dot{\theta}$$
$$y = a\sin(\omega t) - l\cos\theta \qquad \dot{y} = a\omega\cos(\omega t) + l\sin\theta\dot{\theta}$$

(again, the complicated constraint is implicitly encoded here)

Pendulum with a Rotating Support

Now, we calculate T:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2}\Big[a^2\omega^2 \sin^2(\omega t) - 2a\omega l\dot{\theta}\sin(\omega t)\cos\theta + l^2\dot{\theta}^2\cos^2\theta$$

$$+ a^2\omega^2\cos^2(\omega t) + 2a\omega l\dot{\theta}\cos(\omega t)\sin\theta + l^2\dot{\theta}^2\sin^2\theta\Big]$$

$$= \frac{m}{2}\left[a^2\omega^2 + l^2\dot{\theta}^2 + 2a\omega l\dot{\theta}\left(\sin\theta\cos(\omega t) - \cos\theta\sin(\omega t)\right)\right]$$

$$= \frac{m}{2}\left[a^2\omega^2 + l^2\dot{\theta}^2 + 2a\omega l\dot{\theta}\sin(\theta - \omega t)\right]$$

Now, define U=0 at y=0. Then,

$$U = mgy = mg\left(a\sin(\omega t) - l\cos\theta\right)$$

$$= mga\sin(\omega t) - mgl\cos\theta$$

Pendulum with a Rotating Support

Then, $L = T - U$ gives:

$$L = \frac{m}{2}\left[a^2\omega^2 + l^2\dot{\theta}^2 + 2a\omega l\dot{\theta}\sin(\theta - \omega t)\right] - mga\sin(\omega t) + mgl\cos\theta$$

Now, form the EOM, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + ma\omega l\sin(\theta - \omega t)$$

$$\frac{\partial L}{\partial \theta} = ma\omega l\dot{\theta}\cos(\theta - \omega t) - mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + ma\omega l\cos(\theta - \omega t)(\dot{\theta} - \omega)$$

Pendulum with a Rotating Support

Putting them together, we have,

$$ml^2\ddot{\theta} + ma\omega l\cos(\theta - \omega t)(\dot{\theta} - \omega) - ma\omega l\dot{\theta}\cos(\theta - \omega t) + mgl\sin\theta = 0$$

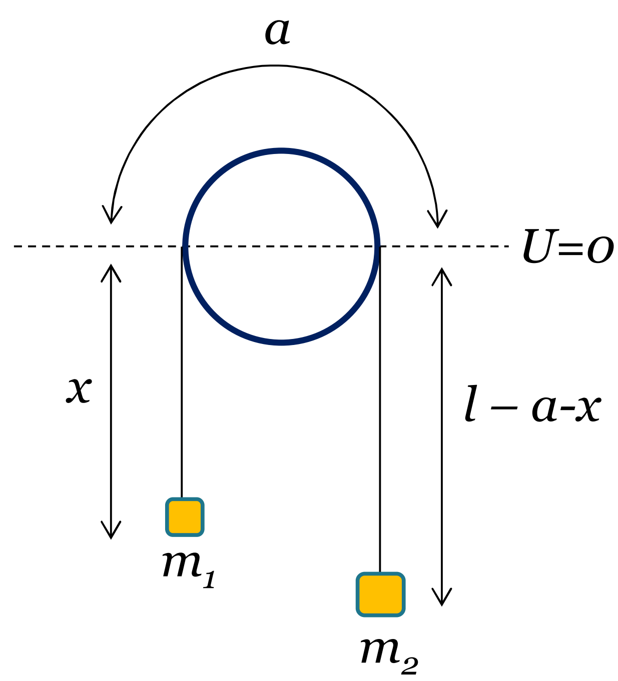
Dividing by ml^2 and simplify, we have,

$$\ddot{\theta} + \frac{a}{l}\omega\cos(\theta - \omega t)(\dot{\theta} - \omega) - \frac{a}{l}\omega\dot{\theta}\cos(\theta - \omega t) + \frac{g}{l}\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta - \frac{a}{l}\omega^2\cos(\theta - \omega t) = 0}$$

(note: if ω =0, we get back the simple pendulum equation.

Atwood Machine



The Atwood Machine is a system with two masses connected by a fixed length string *l* through a pulley.

With a fixed string connecting the two masses, the system is best described by a single generalize coordinate *x*.

$$T = \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2$$

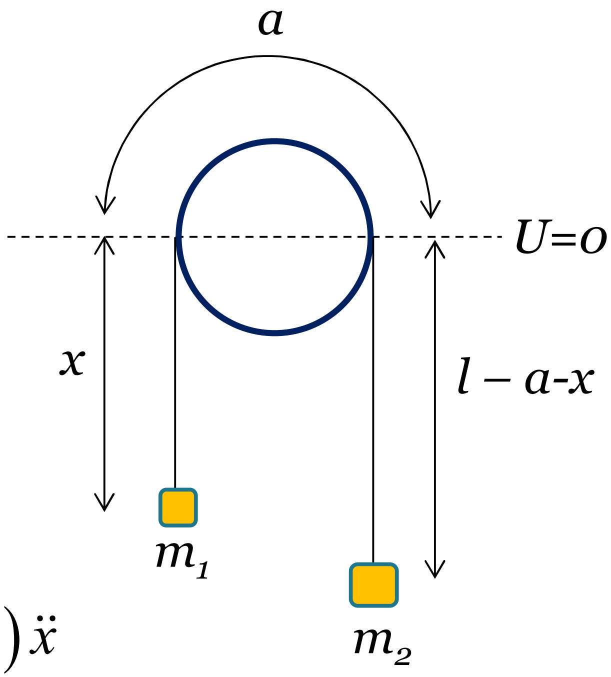
$$U = -m_1 g x - m_2 g\left(l - a - x\right) = -m_1 g x - m_2 g\left(l - a\right) + m_2 g x$$

$$U = \left(m_2 - m_1\right) g x - m_2 g\left(l - a\right)$$

Atwood Machine

$$L = T - U$$

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_2 - m_1)gx + m_2 g(l - a)$$



Now, apply the E-L equation:

$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} \quad \rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_1 + m_2)\ddot{x}$$

$$\frac{\partial L}{\partial x} = -(m_2 - m_1)g$$

E-L Eq $\Rightarrow$ $(m_1 + m_2)\ddot{x} + (m_2 - m_1)g = 0$ or $\ddot{x} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$